Math 249 Wednesday, April 15

$\pi = 51423$

$\boxed{5}\ \boxed{1}\ \boxed{4}\ \boxed{2}\ \boxed{3}$

Permutation $\pi \underset{RSK}{\longleftrightarrow} (P, Q)$ SYT of same shape

$P = j_{\square}(\pi)$ $\qquad Q(\pi) = Q(\pi') \iff \pi \sim \pi'$

$\pi \underset{[i]}{\sim} \pi' \Rightarrow$ same Q, $\quad P \underset{[i]}{\sim} P'$

$i \quad i{+}1 \quad i{+}2$

π —— acb —— $\quad$ or $\quad$ —— bac ——

π' —— cab —— $\qquad\qquad$ —— bca ——

$a<b<c$

$\pi^{-1} \underset{[i]}{\sim} (\pi')^{-1} \Rightarrow$ Same P, $\quad Q \underset{[i]}{\sim} Q'$

Suggests $(P(\pi^{-1}), Q(\pi^{-1})) = (Q(\pi), P(\pi))$. Also reduces proof to doing it for any one π with $P(\pi)$ of shape λ, for each λ.

$$T_0 = \begin{array}{llll} 9 \\ 7 & 8 \\ 5 & 6 \\ 1 & 2 & 3 & 4 \end{array} = P(\pi) \qquad \begin{array}{llll} 6 \\ 4 & 7 \\ 2 & 5 \\ 1 & 3 & 8 & 9 \end{array}$$

$Q(\pi)$

Building λ by adding horizontal strips of sizes $\lambda_\ell, \dots, \lambda_4, \lambda_3, \lambda_2, \lambda_1$

$\pi = \underbrace{9}\ \underbrace{78}\ \underbrace{56}\ \underbrace{1234}$

$P(\pi) = T_0$

$\pi^{-1} = \underbrace{6789}\ \underbrace{45}\ \underbrace{23}\ 1$

$$P(\pi^{-1}) = \begin{array}{llll} 6 \\ 4 & 7 \\ 2 & 5 \\ 1 & 3 & 8 & 9 \end{array} = Q(\pi)$$

$$Q(\pi^{-1}) = \begin{array}{llll} 9 \\ 7 & 8 \\ 5 & 6 \\ 1 & 2 & 3 & 4 \end{array} = T_0$$

Theorem $\pi \mapsto \pi^{-1} \underset{RSK}{\dashrightarrow} (P, Q) \mapsto (Q, P)$

RSK: words $\to (P, Q)$, $\quad P \in SSYT(\lambda)$, $Q \in SYT(\lambda)$

$\underbrace{\text{Multisets of pairs } \binom{i}{j} \in \mathbb{N} \times \mathbb{N}}$ $\xrightarrow[RSK]{} (P, Q)$, $\quad P, Q \in SSYT(\lambda)$

"bipartite partitions"

order lexicographically by j, then i.

$x_1^2 x_2^3 x_3 x_4^3 \quad y_1^3 y_3^2 y_5^4$

Ex:

$$\begin{array}{c|ccccccccc} w & 1 & 4 & 4 & 2 & 4 & 1 & 2 & 2 & 3 \\ z & 1 & 1 & 1 & 3 & 3 & 5 & 5 & 5 & 5 \end{array}$$

'unsorting permutation' for the first row

$\pi \quad 1\ 7\ 8\ 3\ 9\ 2\ 4\ 5\ 6$

π, $wt(w) = \{1^2, 2^3, 3^1, 4^3\} = \alpha$

$wt(z) = \{1^3, 3^2, 5^4\} = \beta$

$$\begin{array}{ccccccccc} 1 & 4 & 4 & 2 & 4 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 3 & 3 & 5 & 5 & 5 & 5 \end{array}$$

$D(\pi) = \{3, 5\}$

$= D(\text{tableau } \pi^{-1})$

$1_1\ 4\ 4\ 2_1\ 4\ 1_2\ 2_2\ 2_3\ 3 \rightarrow 1\ 7\ 8\ 3\ 9\ 2\ 4\ 5\ 6$

z is weakly increasing, but strictly increases at descents of tableau π^{-1}

$\rightarrow wt(z)$ is admissible for tableau π^{-1}

$\rightarrow wt(w)$ is admissible for tableau π

$\pi \xrightarrow{RSK} (P, Q) \qquad SYT(\lambda)$

label with $wt(w)$ — label with $wt(z)$

$(w, z) \overset{RSK}{\longmapsto} (P', Q')$

bipartite partitions $\longleftrightarrow$ pairs of SSYT of same shape

$1\,1\,2\,2\,2\,3\,4\,4\,4$

$$\begin{array}{lllll} 7 \\ 3 & 8 & 9 \\ 1 & 2 & 4 & 5 & 6 \end{array} \qquad \begin{array}{lllll} 6 \\ 4 & 7 & 8 \\ 1 & 2 & 3 & 5 & 9 \end{array}$$

$P(\pi) \qquad Q(\pi)$

$$\begin{array}{lllll} 4 \\ 2 & 4 & 4 \\ 1 & 1 & 2 & 2 & 3 \end{array} \qquad \begin{array}{lllll} 5 \\ 3 & 5 & 5 \\ 1 & 1 & 1 & 3 & 5 \end{array} \qquad \begin{array}{lllll} 4 \\ 2 & 4 & 4 \\ 1 & 1 & 2 & 2 & 3 \end{array} \qquad \begin{array}{lllll} 5 \\ 3 & 5 & 5 \\ 1 & 1 & 1 & 3 & 5 \end{array}$$

$P' \qquad Q' \qquad P'$

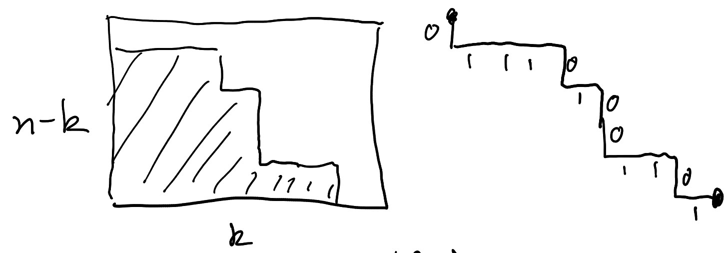
$(w \leftrightarrow z) \leftrightarrow (P, Q)$

$wt(w) = wt(P) \qquad wt(z) = wt(Q)$

Cor. to RSK: $\displaystyle\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$ Cauchy formula!

OGF for bipartite partitions

$[n]_q = 1 + q + \cdots + q^{n-1}$ $\qquad [n]_q! = [1]_q [2]_q \cdots [n]_q$ $\qquad \binom{n}{k}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$



0 1 1 1 0 1 0 0 1 1 1 0 1 $\quad$ k 1's

n-k 0's

n total

# of these is $\binom{n}{k}$.

$$\sum_{\lambda \subseteq (k^{n-k})} q^{|\lambda|} = \binom{n}{k}_q$$

q-analogs in symmetric function theory.

Hall-Littlewood polynomials $\quad H_\mu(x;q) = \sum_\lambda K_{\lambda\mu}(q)\, s_\lambda(x)$ $\qquad H_\mu(x;1) = h_\mu \Rightarrow K_{\lambda\mu}(1) = K_{\lambda\mu}$

$= |SSYT(\lambda,\mu)|$

$K_{\lambda\mu}(q) \in \mathbb{N}(q)$

$= \sum_{T \in SSYT(\lambda,\mu)} q^{ch(T)}$ $\leftarrow$ "charge" (Lascoux-Schützenberger)

$$s_\lambda(x_1,\dots,x_n) = \frac{a_{\lambda+\rho}(x_1,\dots,x_n)}{a_\rho(x_1,\dots,x_n)}$$

$$= \sum_{w \in S_n} w\left(\frac{x_1^{\lambda_1}\cdots x_n^{\lambda_n}}{\prod_{i<j}(1 - x_j/x_i)} \right)$$ $\leftarrow$ Weyl character formula

$$\prod_{i<j}(1 - x_j/x_i) = \underbrace{x_1^{1-n} x_2^{2-n}\cdots}_{x^{-\rho}} \underbrace{\prod_{i<j}(x_i - x_j)}_{a_\rho}$$

$$\sum_{w\in S_n} w\left(\frac{x^{\lambda+\rho}}{a_\rho}\right) = \frac{\sum (-1)^w \cdot w(x^{\lambda+\rho})}{a_\rho} = \frac{a_{\lambda+\rho}}{a_\rho}$$

$$\sum_{w \in S_n} w\left(\frac{x^\lambda}{\prod_{i<j}(1-x_j/x_i) \cdot \prod_{i<j}(1 - q\, x_i/x_j)} \right)$$

"

power series in q, coefficients are symmetric Laurent pols, linear combs of "Schur functions" $s_\mu \quad \mu \in \mathbb{Z}^n$"

as power series in q, has Laurent pol coefficients in $\underline{x}$.

$$\frac{1}{\prod_{i<j}(1 - q x_i/x_j)}$$

$$H_\lambda(x;q) = \left(\sum_{w\in S_n} w\left(\frac{x^\lambda}{\prod_{i<j}(1-x_j/x_i)\prod_{i<j}(1-qx_i/x_j)} \right) \right)_{pol} \leftarrow \text{keep only } s_\mu : \mu \text{ is a partition.}$$

Hall-Littlewood